An Exact, Optimal Strategy for Traditional vs. Roth IRA/401(k) Consumption During Retirement*

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Abstract

In this article we establish an algorithm that exactly determines a strategy for American retirees to optimally withdraw funds between their tax-deferred accounts (TDAs), like traditional IRA/401(k) accounts, and their Roth IRA/401(k) accounts. This optimal strategy follows a geometrically simple, intuitive approach that can be used to maximize the size of a retiree's bequest to an heir or, alternatively, to maximize a retiree's portfolio longevity. We give examples where retirees following the approach currently implemented by major investment firms, like Fidelity and Vanguard, will reduce their bequests by approximately 10% or lose 18 months of portfolio longevity compared to our optimal approach. Further, our strategy and algorithm can be applied to any case where the retiree has additional, known yearly sources of money, such as income from part-time work, taxable investment accounts, and Social Security.

Introduction

In the United States, Roth accounts have grown significantly in popularity since 1997 when the Roth IRA was introduced. This increase in popularity was aided in 2006 by the creation of the financially equivalent Roth 401(k) and, in 2010, by the de facto removal of income limits for Roth IRA contributions by using conversions from traditional IRAs. Indeed, the total amount of Roth IRA assets in America have increased approximately five-fold from 2003 to 2013, while, by comparison, traditional IRA and 401(k) contributions have only roughly doubled during this period, according to the Investment Company Institute (2014). As these more recent Roth vehicles take their place next to Tax-Deferred Accounts (TDAs) like traditional IRAs and traditional 401(k)s, a progressively important question arises: What is the best withdrawal strategy between TDA funds and Roth funds that a retiree can choose?

Previous work has considered and compared the effects of different strategies. Our approach is different in that we present a strategy that is guaranteed to be optimal. Our optimal strategy has a simple geometric interpretation: On a bar graph where projected consumption needs are given for each projected year of retirement, optimal TDA and Roth withdrawal amounts are obtained by letting TDA funds behave "like liquid," sinking to the bottom of the graph to determine the TDA withdrawal amounts, while Roth funds behave "like gas," rising to the top of the bar graph. This can be seen, for example, by previewing Figure 3, where optimal TDA withdrawals are in blue and optimal Roth withdrawals are in green, or Figure 8, where, in addition to the TDA and Roth withdrawals, we have unmet consumption needs in red. Because our algorithm is optimal, any consumption needs that

are unmet by our strategy are impossible to fulfill, so our paper enables financial advisors to know when their clients' projected consumption needs cannot possibly be met with their current resources.

Further, our "liquid-gas" strategy is also optimal when, as in Figure 4, the retiree has additional sources of money other than the TDA that are taxed as income, which are shown in yellow in Figure 4, or additional sources of money other than the Roth that are not taxed as income, shown in magenta in Figure 4. The retiree's goal may be to maximize their bequest or to maximize their portfolio's longevity. Our strategy applies to either goal. It also applies regardless of how much money is in the TDA and Roth accounts.

In Section 1, we summarize previous work on TDA vs. Roth withdrawal strategies. In Section 2, we describe our optimal strategy in more detail, using a key fact demonstrated in Appendix 1 that only tax rates, not timing, determine the effectiveness of a withdrawal strategy. In Section 3, we outline the algorithm that exactly computes our optimal strategy in the case of maximizing bequest size. The details of this algorithm are provided in Appendices 2 and 3. In Section 4, we discuss four categories into which our optimal strategy for maximizing the retiree's bequest can fall, and we show how to adapt our optimal strategy for bequests to find an optimal strategy if the retiree's goal is, instead, to maximize their portfolio's longevity. In Section 5, we consider the effects of Required Minimum Distributions (RMDs) for TDA accounts, both in the context of maximizing bequest size and in the context of maximizing portfolio longevity. Section 6 closes with a summary of our conclusions.

Previous TDA and Roth Consumption Strategies

There appears to be universal agreement that any required minimum distributions (RMDs) from a TDA should be paid. After that, however, there is considerable disagreement about TDA vs. Roth withdrawal sequencing. A number of books on retirement, such as Solin (2010), Rodgers (2009), and Lange (2009) advocate that TDA money be drained before Roth money. This strategy, which is demonstrated in the left panel of Figure 1, is also advocated by Jaconetti and Bruno (2008) from the Vanguard Group and used by Fidelity's Retirement Income Planner (2014) online software. Other books, like Larimore (2011), state that Roth money should be drained before TDA money as shown, ignoring RMD requirements, in the right panel of Figure 1. Following Horan (2006), we refer to these two strategies represented in Figure 1 as "naïve strategies."

Other sources, such as Horan (2006) and Reichenstein et al (2012), disagree with both of these naïve strategies, stating that the order of withdrawals between TDA and Roth accounts is irrelevant if the tax rate is flat. They point out that tax brackets need to be considered, since TDA withdrawals are taxed as ordinary income. Horan (2006) introduces a specific "informed strategy" based on this viewpoint. Under the assumption that RMDs are satisfied, Horan's (2006) "informed strategy" drains TDA money up to the top of a fixed tax bracket and then uses Roth money thereafter. The best informed strategy uses the top of the specific tax bracket that leads to the greatest portfolio longevity. This viewpoint is represented in the left panel of Figure 2 under the context of optimizing the

bequest to an heir. Note that Horan (2006) makes the historically accurate assumption that tax brackets are automatically adjusted by a measure of inflation, which is reflected in the figure, and will also be assumed throughout this paper. This informed strategy model is advocated in the book by Piper (2013) and is further reinforced by Sumutka et al (2012) in the context where taxable stock accounts are also considered under a comprehensive tax model.

We establish in Appendix 1 that the sequencing concerns addressed by the naïve strategies do not exist under a constant tax rate, confirming the viewpoint of Horan (2006) and Reichenstein et al (2012). In particular, we show that if there is a flat tax rate for all TDA withdrawals, then both of the naïve strategies result in portfolios with equal longevity. This means that tax minimization is the *only* consideration for optimally sequencing TDA and Roth withdrawals. The informed strategy moves considerably towards this direction, but is generally suboptimal due to requiring that withdrawals reach the top of a tax bracket. Usually, this either produces a situation as in Figure 2 where no TDA money is being used in the final years of the portfolio, thereby failing to take advantage of the lowest tax brackets in those years, or it produces a situation where TDA money remains after the Roth money is exhausted, so the TDA money is either unused or used in unnecessarily high tax brackets. These problems with the informed strategy can occur whether the goal is to optimize the size of a bequest or to optimize a portfolio's longevity.

An Optimal Strategy with a Simple Geometric Framework

For the remainder of this paper, we will only use real dollars based on the initial time $t = 0$,[1] where the conversion from nominal to real dollars uses the rate of inflation assumed for the tax brackets. Using real dollars has the distinct advantage of making the tax brackets constant over time. The left panel of Figure 2 is in nominal dollars, while the right panel of Figure 2 is in real dollars, using the same informed strategy, and has tax brackets that are constant over time. Because the tax brackets are constant if we use the real dollar perspective, we see that we can minimize taxes by making TDA withdrawals that are constant in real dollars over time, as shown in Figure 3. Table 1 summarizes the terminal wealth after taxes corresponding to the strategies that appear in Figures 1–3, demonstrating the advantage of using an exact optimal strategy that makes constant TDA withdrawals, in real dollars, during retirement.

The optimal strategy presented in Figure 3 follows a simple analogy between the physical properties of liquids and gases. The annual consumption needs can be thought of as interconnected columns that can be filled with either liquid or gas. TDA consumption fills these cylinders like liquid, since liquid falls to the bottom of these interconnected columns and attains a uniform level. Roth consumption fills these cylinders like gas, rising as high as it can go in the interconnected columns. The gas attains a uniform level at its bottom, just as the liquid attains a uniform level at its top. Any space between these two levels corresponds to unmet consumption.

[1] The one exception to this will be in Appendix 2, where the pseudocode is made clearer by basing the real dollars on the final time when the retiree dies instead of $t = 0$.

This "liquid-gas" viewpoint easily extends to provide an optimal strategy in circumstances where the retired investor has additional known sources of retirement income. We divide these additional sources into two categories. The first category includes all sources of money other than the Roth account that are not subject to income tax. These include long-term capital gains, qualified dividends, some parts of Social Security, and some pensions and annuities. After any taxes are removed, these can be used to satisfy the retiree's consumption needs and so, on an after-tax basis, this money reduces the height of the consumption columns in the years it is received. From a physical perspective, this category of money lowers the "ceiling" on which the Roth "gas" will stop as it rises.

The second category encompasses all sources of money other than the TDA account that are subject to income tax. These include earned income, short-term capital gains, unqualified dividends, some parts of Social Security, and some pensions and annuities. The effect of this category may be ameliorated through the use of capital losses and other deductions. Because these sources affect the tax brackets that the additional TDA money will use, we address this money, again on an after-tax basis, by filling the bottom of the consumption columns with it. From a physical perspective, we can think of this money as acting like a solid sand bed at the bottom of the columns. We then fill as before: the TDA acts like liquid, reaching a uniform height above the sand bed, and the Roth acts like gas, filling up the space above the liquid and any parts of the sand bed that lie above the TDA "liquid line." Figure 4 demonstrates optimal TDA and Roth consumption in the presence of both categories of additional income, where the magnitudes of both categories change significantly over time. This demonstrates how our "liquid-gas" optimal strategy, now extended to a "solid sand bed-liquid-gas-lowered ceiling" viewpoint, works, even under an extreme circumstance.

The goal of maximizing the bequest to an heir is far less studied than the goal of maximizing portfolio longevity. We will, at first, focus on maximizing the bequest, similar to work by Al Zaman (2008). We will later show how the method of maximizing the bequest is easily extended to attain the more studied goal of maximizing portfolio longevity.

Determining and Computing and Optimal Consumption Strategy

Assumptions. We make a common set of assumptions to analyze optimal strategies in retirement. Specifically, we assume

1. We know the TDA and Roth balances prior to the first year, $t = 1$.
2. For each year, t, of our model we know...
    - $C(t)$, the retiree's (after-tax) consumption needs in real dollars.
    - $U(t)$, after-tax money in real dollars available in year t coming from sources other than the Roth account that are not subject to income tax.
    - $L(t)$, after-tax money in real dollars available in year t coming from sources other than the TDA account that are subject to income tax. By these definitions, all sources of money available to the retiree are categorized as being part of the Roth account, the TDA account, $U(t)$, or $L(t)$.

- $\mu(t)$, the real rate of return from both capital gains and dividends in year t for both the TDA and Roth accounts. For simplicity, we will use cases where $\mu(t)$ is constant in this paper, however, our model can easily accommodate any given function $\mu(t)$.
- The tax brackets, which are assumed to be constant in real dollars over time, and the corresponding tax rates of each of these brackets. In our figures, we have used the 2014 tax brackets and rates published by the IRS for single filers.

3. We know t_{death}, the year in which the retiree dies, and we know τ_{heir}, the marginal tax rate (or effective marginal tax rate) of the heir. Note that if the "heir" is a tax-exempt charity, then $\tau_{\text{heir}} = 0$.
4. Finally, we assume that the inheritance is not so high that the inheritance tax is relevant, and we assume that our TDA withdrawals are sufficiently large to satisfy RMD requirements. We will discuss this RMD assumption in more detail below, including showing why this assumption is unlikely to be violated if the goal of the retiree is for the portfolio to last throughout their lifetime.

Summary of our Optimal Strategy to Withdraw from a TDA Account and a Roth Account. If the retiree's TDA and Roth accounts are big enough for bequests from both accounts to occur, then the optimal strategy is for the retiree to consume TDA money at all of the marginal tax rates τ_i where $\tau_i \leq \tau_{\text{heir}}$. In the case where $\tau_i = \tau_{\text{heir}}$, there is no tax advantage to having the retiree vs. the heir consume the TDA, just as there is no tax advantage to having the retiree vs. the heir consume the Roth account. However, in this case, consuming TDA money before Roth money reduces the chance of RMDs being an issue for the retiree. After this, the retiree should fill all remaining consumption needs not satisfied by $L(t)$ and $U(t)$ with Roth money.

Should the TDA account be exhausted before satisfying all consumption needs where $\tau_i \leq \tau_{\text{heir}}$, the Roth account should be tapped to satisfy all remaining consumption needs. On the other hand, if the TDA account has filled the consumption needs where $\tau_i \leq \tau_{\text{heir}}$, and then the Roth account becomes exhausted while attempting to meet the consumption needs where $\tau_i > \tau_{\text{heir}}$, then the TDA account must again be tapped to fill the remaining consumption needs. We next describe an algorithm that elaborates on how to attain this optimal strategy by following the "liquid-gas" viewpoint previously described.

An Exact Algorithm to Optimally Withdraw from a TDA Account and a Roth Account. We decompose the retiree's after-tax consumption needs

$$C(t) = U(t) + M(t),$$

where $M(t)$ is the after-tax consumption needs to be fulfilled by Roth money, TDA money, or other sources of income that are subject to income tax, $L(t)$. We think of columns of height

$M(t)$ connected in chronological order from $t = 1$ to $t = t_{\text{death}}$.

Define $i\text{max}$ to be the number of tax brackets, and, for each tax bracket $i = 1, 2, ..., i_{\text{max}}$, define τ_i to be the marginal tax rate and define H_i to be the after-tax income (i.e., height) corresponding to the top of the ith tax bracket. For example, for single filers in 2014, the lowest tax bracket is 10%, which covers before-tax income between \$0 and \$9,075. \$9,075 corresponds to an after-tax income of $.9 * \$9{,}075 = \$8{,}167.50$, and so $\tau_1 = 0.10$ and $H_1 = \$8{,}167.50$. The next tax bracket is 15% on before-tax income between \$9,075 and \$36,900, so $\tau_2 = 0.15$ and $H_2 = 8167.5 + .85 * (36900 - 9075) = \$318{,}18.75$. Note that the horizontal dashed lines in the figures of this paper correspond to H_1, H_2, etc. The horizontal solid line in each figure corresponds to the critical height $\underline{H}_{\text{heir}}$, which we define to equal $H_{i,\text{heir}}$, where i_{heir} is the largest value of i where $\tau_i \leq \tau_{\text{heir}}$.

To attain our optimal strategy, ideally we use TDA money to fill the columns up to any necessary "liquid level" at or below H_{heir}. Once the level H_{heir} is reached, we fill the remaining consumption needs in the columns using Roth "gas," which rises to the top of the connected columns. If we run out of either "liquid" or "gas" at any point, then we must then use the remaining fund to satisfy all further consumption needs. If both funds are exhausted before the consumption needs are filled, it is impossible for the retiree to meet their desired consumption level, C(t).

Specifically, we fill the columns of height $M(t)$ in, at most, the following four steps:

- *Step 1: Fill with income other than the TDA that is subject to income tax.* We first fill the columns corresponding to each time t with sand up to the height $L(t)$. The after-tax dollar distance in a column from the top of the sand to the top of the column, $M(t) - L(t)$, corresponds to the after-tax consumption needs at the column's time, t, that must be satisfied by consuming TDA and Roth money.
- *Step 2: Fill using TDA money that is subject to marginal income tax rates that are less than or equal to the projected effective marginal rate of the heir.* We fill the columns with TDA "liquid" until the TDA account is exhausted, the columns are all filled, or h, the level of the liquid in the unfilled columns, reaches the height H_{heir}. See Appendix 2 for details about this part of the algorithm. Note that the TDA liquid may completely fill some columns with smaller values of $M(t)$, in which case any additional liquid then goes strictly to filling the columns that are not yet full. If all the columns are completely filled with TDA money after this step, in which case $M(t) \leq H_{\text{heir}}$ for all t, all consumption needs are optimally met and the algorithm is finished. Otherwise, there is at least one unfilled column, in which case we proceed to step 3.
- *Step 3: Fill using Roth money.* If TDA money was exhausted in step 2, then we must tap the Roth money now. If TDA money was not exhausted after step 2 and the retiree were to continue to use the TDA money, then $h > H_{\text{heir}}$, which corresponds to a higher tax rate for the retiree than for the heir. It is better, therefore, to first drain the tax-free Roth account instead, since it makes no difference whether the retiree or the heir uses the Roth account from a tax point of view.

  The Roth "gas" fills the columns as gas would fill them, rising to the top of the columns. In Appendix 3, we explain that the algorithm for this gas filling is similar to the liquid filling in step 2 with the entire column system flipped upside-down. If all the columns are full after this step, then all consumption needs are optimally met and the algorithm is finished.

  Otherwise, the Roth money is exhausted before all the columns could be filled. If we

also exhausted the TDA money in step 2, it is impossible for the retiree to satisfy the desired consumption levels, $C(t)$, so the retiree must reduce $C(t)$ and rerun the algorithm. But if TDA money is still available, we proceed to step 4.

- *Step 4: Fill using remaining TDA money.* We continue to fill the columns with TDA "liquid" until they are all full or all the TDA money is exhausted. The algorithm here is essentially the same as that in step 2. If we succeed in filling all the columns, the algorithm is finished. If we exhaust the TDA money before filling all the columns, it is again impossible for the retiree to satisfy the desired consumption levels, $C(t)$, so the retiree must reduce $C(t)$ and rerun the algorithm.

Results

Strategy for Optimal Bequest. We look at some examples of strategies generated by our algorithm to optimize the after-tax bequest. We choose consumption needs, $C(t)$, that increase over time in real dollars to model an increase in the retiree's medical costs above the general rate of inflation. There are four possible cases for the optimal strategy generated by our algorithm, corresponding to whether or not TDA money is left to the heir and whether or not Roth money is left to the heir. We label these four cases a–d. Examples for cases a–d are given in Figures 5–8.

In Figure 5, we have an example of "case a," where both TDA money and Roth money are left to the heir. For all examples of case a, step 2 of our algorithm ends with the TDA level rising to H_{heir}, step 3 ends with the Roth level coming down to H_{heir}, and step 4 does not happen.

In Figure 6, we have an example of "case b," where the heir gets Roth money but no TDA money. For case b, the retiree exhausts their TDA money in step 2, and the rest of the consumption needs are filled by Roth money in step 3. As in case a, step 4 does not happen.

Figure 7, which corresponds to "case c," is the opposite situation of case b. Here, the heir gets TDA money but no Roth money. In this case, the TDA money fills to the level H_{heir} in step 2, and then the Roth money is exhausted in step 3, requiring TDA money to be used in step 4 to meet the remaining consumption needs.

In cases a–c where there is money for the heir, the liquid level of the TDA is equal to the level of the bottom of the Roth gas. Call this height level h. In cases b or c, as we increase μ, the real rate of return for both the TDA and Roth accounts, h moves closer to H_{heir}, and once $h = H_{\text{heir}}$, we cross over to case a, where both TDA and Roth money are given to the heir.

Finally, in Figure 8, the retiree runs out of both TDA and Roth money, which is "case d." The retiree can run out of TDA money in either step 2 or step 4. For the case in Figure 8, the retiree runs out in step 2. Either way, Roth money runs out in step 3. The gap in red between the level of the TDA liquid and the level of the Roth gas represents the unmet consumption. Because our algorithm is optimal, there is no possible way to satisfy this, so the retiree has no choice but to reduce consumption and/or find additional sources of income.

What is the effect of decreasing μ, the investment rate of return, on the optimal

strategy? If we are in case a, where $h = H_{\text{heir}}$, and then reduce μ, we will eventually run out of either bequeathed TDA money or bequeathed Roth money. If we run out of bequeathed TDA money first, we transition into case b, and h will decrease. If we run out of bequeathed Roth money first, we transition into case c, and h will increase. Eventually, we will run out bequeathed money for both accounts, and transition into case d with an unmet consumption gap. Note that for these case transitions to occur, μ may need to be decreased to the point where it becomes negative, which corresponds to portfolio losses instead of gains in real dollars.

Strategy for Optimal Portfolio Longevity. We can easily adapt the algorithm for the optimal bequest to determine optimal portfolio longevity, because optimal portfolio longevity occurs at the exact point when cases b or c transition to case d. In other words, the point when we run out of retirement funds. To find this point in time, we alter t_{death}, in a manner similar to our altering μ in the previous paragraph.

More specifically, we first run our algorithm using an arbitrary value of t_{death}. If we find we are in case d, we reduce t_{death}; if we find we are in cases a–c, we increase t_{death}. We can accommodate a fraction α of year t_{death}, where $0 \leq \alpha < 1$, by using $\alpha C(t_{\text{death}}), \alpha U(t_{\text{death}})$, and $\alpha L(t_{\text{death}})$ in place of $C(t_{\text{death}}), U(t_{\text{death}})$, and $L(t_{\text{death}})$, noting that this will reduce the size of $M(t_{\text{death}})$ to $\alpha M(t_{\text{death}})$. The value of t_{death} will converge to t_{opt}, which is the optimal portfolio longevity. By considering values of t_{death} that increase to t_{opt}, we will generate values of h that converge to h_{opt}, yielding an optimal strategy for portfolio longevity.

The value of H_{heir} chosen to run the algorithm for optimal portfolio longevity is irrelevant to the end result, since there are no retirement funds given to the heir. As t_{death} increases to t_{opt}, the corresponding values of h that are generated will decrease to h_{opt} if $H_{\text{heir}} > h_{\text{opt}}$ and the values of h will increase to h_{opt} if $H_{\text{heir}} < h_{\text{opt}}$.

As a basic example, consider the optimal longevity determined in Figure 9. The scenario for this figure has the TDA account start with \$290,000 and the Roth account start with \$460,000 dollars. Both accounts grow at a real annual rate of 5%. The consumption needs of the retiree remain constant at \$41,000 real dollars per year. We choose τ_{heir} = 33%, which corresponds to H_{heir} = \$140,996, for the runs, although this choice is irrelevant to the optimal longevity that we determine, as previously noted.

The upper left panel in Figure 9 assumes t_{death} = 30 years, which is less than t_{opt}, since it leaves almost \$225,000 real ($t = 0$) Roth dollars as a bequest. In contrast, the lower right panel in Figure 9 assumes t_{death} = 40, which is greater than t_{opt}, since it generates a red band corresponding to \$1,333 of unmet consumption need every year. The upper right panel assumes t_{death} = 36 years, which is slightly below t_{opt}, leaving just below \$12,000 real Roth dollars as a bequest, while the lower left panel assumes t_{death} = 37, which is slightly higher than t_{opt}, corresponding to about \$283 of unmet consumption every year.

Therefore, the optimal portfolio longevity is between 36 and 37 years. By shifting the value of α, we find that the optimal α is 0.30, so the optimal portfolio longevity is 36.30 years. This corresponds to the middle panel in Figure 9, which also gives our

optimal portfolio strategy: spending \$26,500 (real) after-tax TDA dollars and \$14,500 (real) Roth dollars in each of the first 36 years and then satisfying the consumption needs in the 37th year solely with TDA money.

In Table 2, we compare our optimal portfolio longevity for the scenario used in Figure 9 to the portfolio longevity obtained by using the two naïve strategies and the best informed strategy.

Our exact optimal longevity strategy can also be applied when $L(t)$ and/or $U(t)$ are non-zero. An example of this is presented in Figure 10. In this example, $L(t)$ represents part time work income that decreases during retirement. $U(t)$ represents an annuity with a constant (in nominal dollars) yearly payout, not subject to income tax, such as what might be obtained from a defined benefit plan.

Required Minimum Distributions (RMDs)

RMDs When Optimizing Bequests. RMDs may be an obstacle when the retiree's goal is to optimize their bequest. In particular, if H_{heir} is low, then our optimal strategy suggests reducing TDA payments, which may conflict with RMD rules. In the extreme case of a charitable bequest where $H_{\text{heir}} = 0$, RMDs are guaranteed to be an issue, since the retiree ideally only uses Roth money in order to minimize taxes. In this extreme case, should the retiree have sufficient Roth money, the optimal strategy is to use TDA money to satisfy RMDs and then use Roth money for all remaining consumption needs.

When the optimal strategy generated by our algorithm does not satisfy RMDs and we are either in case b or c, there may be alternate optimal strategies that can avoid these RMD issues. By "optimal strategy," we mean any strategy that satisfies all consumption needs and leads to the same bequest as the strategy generated by our previously presented optimal algorithm in the absence of RMDs. We begin by specifying the complete set of optimal strategies.

Define $h(t)$ to specify in each year t the level of after-tax consumption at which we switch from TDA spending to Roth spending. This fits with the previous definition of the constant height h generated by the algorithm in cases a–c. $h(t)$ just extends this definition to allow h to vary with time.

In cases b or c, the value of h produced by the algorithm corresponds to a specific tax bracket that contains it. In the figures, this tax bracket corresponds to the horizontal interval that lies between the dashed lines just above and below the horizontal line at height h. Any strategy, $h(t)$, that remains in this tax bracket, (which corresponds to staying within the horizontal interval), satisfies all consumption needs and still exhausts the account (TDA or Roth) that was exhausted with our optimal strategy will be an optimal strategy. All other $h(t)$ are suboptimal, because leaving the interval leads to a tax strategy that is inferior to our optimal strategy.

It may be possible to avoid RMDs by using one of these other optimal strategies. In years where RMDs are an issue, $h(t)$ should be increased to satisfy the RMDs. This increase will force $h(t)$ to be decreased in other years. For one of these optimal strategies to be successful, the RMDs cannot force the retiree into a higher tax bracket (above the horizontal interval) in any year, and the compensating decrease in $h(t)$ has to be accomplished without

creating new RMD violations and without moving the retiree into a lower tax bracket (below the horizontal band) in any later year.

Note that in case a, h lies between tax brackets (see, for example, Figure 5, where h lies between the 25% and 28% tax brackets), so there is no room for $h(t)$ to vary. In other words, the optimal strategy generated by our algorithm in case a is the only optimal strategy.

The Unlikelihood of RMDs When Optimizing Portfolio Longevity. RMDs will rarely pose a problem for retirees who are interested in using our optimal strategy to maximize their portfolio's longevity. To explain why, we first assume for simplicity a reasonably standard case where our optimal strategy has a constant withdrawal rate in real dollars from the TDA. This assumption holds if, for example, $L(t)$, the taxable income other than the TDA, is constant, and the consumption needs, $M(t) > h_{\text{opt}}$ for all $t = 1, 2, \ldots, t_{\text{death}}$, where $t_{\text{death}} = t_{\text{opt}}$.

Next, we make the unrealistically conservative assumption that μ, the real expected return for the TDA and Roth accounts, is zero. The IRS Uniform Life Table from 2013 states that a retiree who is 70, which is the first year RMDs apply, has a distribution period of 27.4 years. Further, the life table has the property that the sum of the retiree age and the IRS distribution period increases over time, which means that the RMDs in the year the retiree turns 70 must decrease in later years. In other words, if RMDs are satisfied by our optimal strategy in the year when the retiree is 70, they will be satisfied in all later years if our optimal strategy continues to be followed.

We can therefore conclude that RMDs will not be an issue for a 70 year old with an optimal portfolio longevity under 27.4 years[2]. This encompasses most Americans near retirement, since most of these Americans have retirement accounts that are significantly underfunded[3]. Also, the smaller group of Americans that have overfunded their accounts will generally have a goal of optimizing their bequest instead of optimizing their portfolio's longevity.

Further, in the more realistic case where the real expected return $\mu > 0$, RMDs become even less of an issue. This is because, for a given $t_{\text{death}} = t_{\text{opt}}$, the larger the value of μ used in the model, the further TDA money that is unused at age 70 will stretch to satisfy consumption in later years. This means that the fraction of money used at age 70 is higher when $\mu > 0$ than when $\mu = 0$, which implies the likelihood of having any RMD issues is lower. Therefore, since μ is generally positive, many cases where the optimal portfolio longevity is past the age of 97.4 will still have no RMD issues.

[2] According to 2013 IRS tables, the chance that a 70 year old American will live past the age of 97 is 3.8% for men and 8.6% for women.

[3] See, for example, Greenhouse (2012), Marte (2014), and Siedle (2013). These are news articles published, respectively, in the New York Times, the Washington Post, and Forbes that reference a number of studies documenting this problem.

Conclusions

- We have shown that there is no inherent advantage in sequencing consumption from TDA accounts before or after Roth accounts if a retiree pays taxes under a flat tax rate system and there are no RMD issues. This contradicts the advice of many books in the popular press and major financial institutions, which advocate consuming from one account before the other, but it agrees with the viewpoint expressed in Horan (2006) and Reichenstein et al (2012). The variation in income tax rates, therefore, becomes the sole consideration in optimizing TDA vs. Roth withdrawals for retirees.
- We have shown there is an optimal withdrawal strategy that is remarkably simple from a geometric perspective: The TDA should be drained each year to satisfy consumption up to a constant value over time in real dollars for the total after-tax withdrawals from accounts subject to income tax. We provide an algorithm that calculates the exact value of this constant. Further, our algorithm applies regardless of how much money is in the TDA and Roth accounts.
- Because this simple optimal withdrawal strategy is also robust, we demonstrated that it can be applied to a wide variety of circumstances. It can be used when the retiree's goal is to optimize portfolio longevity, and it can also be used when the retiree's goal is to optimize their bequest to an heir or charity. It can be applied to any starting balances for the TDA and Roth accounts. Further, it accommodates cases where the retiree has other known sources of income, no matter how those sources are taxed.
- We have given examples showing that retirees who follow the best informed strategy, as opposed to our optimal strategy, can lose 8% of their bequest or reduce the longevity of their portfolio by 6 months. Further, these examples show that the na¨ıve approach currently implemented by major investment firms can decrease the bequest size by approximately 10% or reduce portfolio longevity by 18 months.
- The strategy presented in this paper assumes that RMDs are not binding. If they are binding, but not in too severe a way, we have shown how we may be able to adapt our simple, optimal algorithm to find a less simple, but still optimal, strategy that avoids these RMD obstacles.
- In this paper, we have exclusively focused on cases where the retiree's yearly sources of income, other than from the TDA and Roth accounts, are known ahead of time. However, our results are also useful for analyzing cases where the withdrawal amounts from these additional sources are not known and therefore need to be optimized along with the TDA and Roth withdrawal amounts. In most of these more complicated optimization scenarios, software is employed to try to find an optimal withdrawal strategy. The software can fail, however, due to having too many variables to optimize. In particular, for n accounts, there are $(n-1)\ t_{\text{death}}$ variables to optimize. Employing the method presented here removes t_{death} of these variables by quickly and exactly determining the optimal amount of TDA vs. Roth money to use. This makes the software faster, more accurate, and, most importantly, improves its chance of successfully finding an optimal strategy in the context of many accounts. In a follow-on study, we will apply this approach to optimal spending choices from taxable accounts in addition to TDA and Roth accounts.

Appendix 1: The Equivalence of the Two Naïve Strategies Given a Constant Tax Rate

Assume we start at $t = 0$ with T_0 in TDA money subject to a tax rate τ and R_0 in Roth money. Assume we continuously consume at a rate of C after-tax dollars per year. Assume both accounts grow at a constant annual rate, μ. First consider continuously draining the Roth followed by the TDA. The differential equation for draining the Roth is

$$\frac{dR}{dt} = \mu R - C$$

subject to the initial condition $R(0) = R_0$. Separating variables and integrating, we have that t_1, the time at which the Roth account is drained, is

$$t_1 = \int_0^{t_1} dt = \int_{R_0}^{0} \frac{1}{\mu R - C} dR = \frac{1}{\mu} ln\left(\frac{-C}{\mu R_0 - C}\right)$$

During this time, the TDA account grows to $T_0 e^{\mu t_1} = \frac{-T_0 C}{\mu R_0 - C}$. This is the initial condition to the differential equation for draining the TDA, which is

$$\frac{dT}{dt} = \mu T - \frac{C}{1-\tau}.$$

Separating variables and solving, we get that t2, the time to drain the TDA, is

$$t_2 = \frac{1}{\mu} ln\left(\frac{-C/(1-\tau)}{\mu \frac{-T_0 C}{\mu R_0 - C} - C/(1-\tau)}\right).$$

Adding t_1 and t_2, we get that the combined time to drain both accounts is

$$t_1 + t_2 = \frac{1}{\mu} ln\left(\frac{-C}{\mu(1-\tau)T_0 + \mu R_0 - C}\right). \qquad (1)$$

Next consider continuously draining the TDA followed by the Roth. By the same logic, the time t_1 to drain the TDA is

$$t_1 = \frac{1}{\mu} \ln\left(\frac{-C}{\mu(1-\tau)T_0 - C}\right),$$

and the subsequent time to drain the Roth is

$$t_2 = \frac{1}{\mu} ln\left(\frac{-C}{\mu \frac{-R_0 C}{\mu(1-\tau)T_0 - C} - C}\right),$$

which sums to the same total time given in equation (1).

Appendix 2: Pseudocode for Step 2

% Initialization:
$l(t) = L(t)$ for $t = 1, 2, \ldots, t_{\text{death}}$ (see note 1 below)
$T = T_0(1+\mu)^{t_{\text{death}}}$ (see note 2)
$R = R_0(1+\mu)^{t_{\text{death}}}$ (also see note 2)

% Define the function $F(x)$:
$F(x) = \sum_{t=1}^{t_{death}} \max\{0, \left(\min\{M(t), x\} - l(t)\right)(1+\mu)^{t_{death}-t+1}\}$ (see note 3)

% Main algorithm for filling with TDA money up to a maximum of H_{heir}:

For $i = 1$ to i_{heir}
  If $F(H_i) \le T(1-\tau_i)$ (see note 4)
    $T = T - F(H_i)\frac{1}{1-\tau_i}$ (see note 5)

    $l(t) = \max\{l(t), H_i\}$ for $t = 1, 2, \ldots, t_{\text{death}}$ (see note 6)

    If $M(t) \le H_i$ for every $t = 1, 2, \ldots, t_{\text{death}}$ (see note 7)
      $h = \max_t M(t)$
      Stop the algorithm.
  Else (see note 8)
    Define the heights $h_1, h_2, \ldots, h_K$ according to note 9.
    For $k = 1$ to K
      If $F(h_k) \le T(1-\tau_i)$ (see note 10)
        $T = T - F(H_k)\frac{1}{1-\tau_i}$

        $l(t) = \max\{l(t), h_k\}$ for $t = 1, 2, \ldots, t_{\text{death}}$
      Else (see note 11)
        Define the set T to be the set of all times t where $l(t) < M(t)$. (see note 12)

        $h = \frac{T(1-\tau_i)+\sum_{t\in T} l(t)(1+\mu)^{t_{death}-t+1}}{\sum_{t\in T} l(t)(1+\mu)^{t_{death}-t+1}}$ (see note 13)

        $T = 0$ (see note 14)
        $l(t) = \max\{l(t), h\}$ for $t = 1, 2, \ldots, t_{\text{death}}$
        Proceed immediately to step 3.
$h = H_{\text{heir}}$ (see note 15) Proceed to step 3.

Notes:

1. $l(t)$ will represent the evolving bottom as we fill with TDA liquid. We begin with the $L(t)$ sand and after filling a portion with TDA liquid, we then think of this portion as sand when we move to the next step.
2. T and R are the amount of TDA (before taxes) and Roth money given to the heir in t_{death} dollars. We initialize both to be their respective values if the retiree spends no TDA or Roth money at $t = 1, 2, \ldots, t_{\text{death}}$. Here, T_0 and R_0 represent the worth of the TDA and Roth accounts

at the end of $t = 0$.

3. $F(x)$ represents the after-tax consumption above level l up to height x in t_{death} dollars.
4. In this case there is enough TDA money to be able to fill up to the top of the ith tax bracket, which we do.
5. Since we are using TDA money to fill up to height H_i, this money is no longer available to the heir. This step removes that money from the heir's amount, T. Since T is in before-tax t_{death} dollars and $F(H_i)$ is in after-tax t_{death} dollars, we must divide $F(H_i)$ by $(1 - \tau_i)$.
6. This updates $l(t)$. This line of pseudocode essentially turns any currently used TDA money/liquid into sand, in preparation for the next step of filling with more TDA money/liquid.
7. If $M(t) \leq H_i$ then all remaining consumption can be filled with TDA money. We do this by setting h equal to the maximum value of $M(t)$, and then terminate the algorithm, since there is no need for any Roth money.
8. In this case there is not enough TDA money to be able to fill up to the top of the ith tax bracket, so we need to determine where to stop within this tax bracket.
9. Define $H_0 = 0$. The heights $h_1, h_2, \ldots, h_{K-1}$ are then defined as the set of all values of $M(t)$ or $l(t)$ that lie strictly between H_{i-1} and H_i, arranged in increasing order. Finally, h_K is defined to equal to H_i.

   At each h_k where $k = 1, 2, \ldots, K-1$, we either start or stop filling one of the columns with TDA money/liquid. Define $h_0 = H_{i-1}$. For the next part of the algorithm to work correctly, we can only apply it to one (h_{k-1}, h_k) interval at a time, which we now do.
10. In this case there is enough TDA money to be able to fill up to the top of the (h_{k-1}, h_k) interval, i.e., up to the height h_k, which we then do.
11. In this case there is not enough TDA money to be able to fill up to the top of the (h_{k-1}, h_k) interval, so we must determine where in this interval we run out of TDA money.
12. This defines the set of columns where we are filling with TDA money/liquid.
13. This is the value of h that satisfies $F(h) = T(1 - \tau_i)$. In other words, it is the height that corresponds to exhausting the TDA account.
14. Since the TDA account is exhausted, there is no TDA money to bequeath the heir, and we look to see if the remaining consumption can be satisfied with Roth money in step 3.
15. To get to this line of the algorithm, there were sufficient TDA funds to be able to fill up to H_{heir}. Since it is then preferable to use Roth money to TDA money to satisfy further consumption needs, we proceed to step 3 where we use Roth money.

Appendix 3: Algorithm for Step 3

The algorithm for step 3 is a straightforward adaptation of the algorithm for step 2 in Appendix 2. The key observation is that the Roth gas behaves just like the TDA liquid if you turn the system of columns upside down.

To do this we define $M_{max} = \max_t M(t)$ and then flip the column system upside down, so that what was $M_{\max}$ is now at $M = 0$ and what was $M = 0$ is now at $M_{\max}$. This means that what was the old sand bed, $l(t)$, now plays the role of $M(t)$, the heights of the columns, and vice versa. Specifically, the new $M(t)$ equals the old $M_{\max} - l(t)$, and the new $l(t)$ equals the old $M_{\max} - M(t)$.

We then fill with Roth money in place of TDA money. So, for example, R is reduced instead of T being reduced. Note that because there are no relevant tax brackets for Roth money, the algorithm is simplified. At the end of this step 3 algorithm, there may or may not be Roth money remaining. If there is Roth money remaining, then the Roth money used in step 3 rose to the level $M_{\max} - H_{\text{heir}}$, which means all consumption needs are now filled and the algorithm is finished. If there is no Roth money remaining at the end of step 3, it was exhausted at some intermediate stage in the step, in which case, we do the following: First, we define h_{Roth} so that $M_{\max} - h_{\text{Roth}}$ is the level at which the Roth money, viewed as liquid in this upside down system, is exhausted. Second, we flip the columns right-side up again. Third, we redefine $M(t) = \min\{M(t), h_{\text{Roth}}\}$. Fourth, we proceed to step 4, where we look to fulfill the remaining consumption needs with TDA money starting at the height $h = H_{\text{heir}}$ and hoping to fill to the height h_{Roth}, which would mean satisfying the remaining consumption need.

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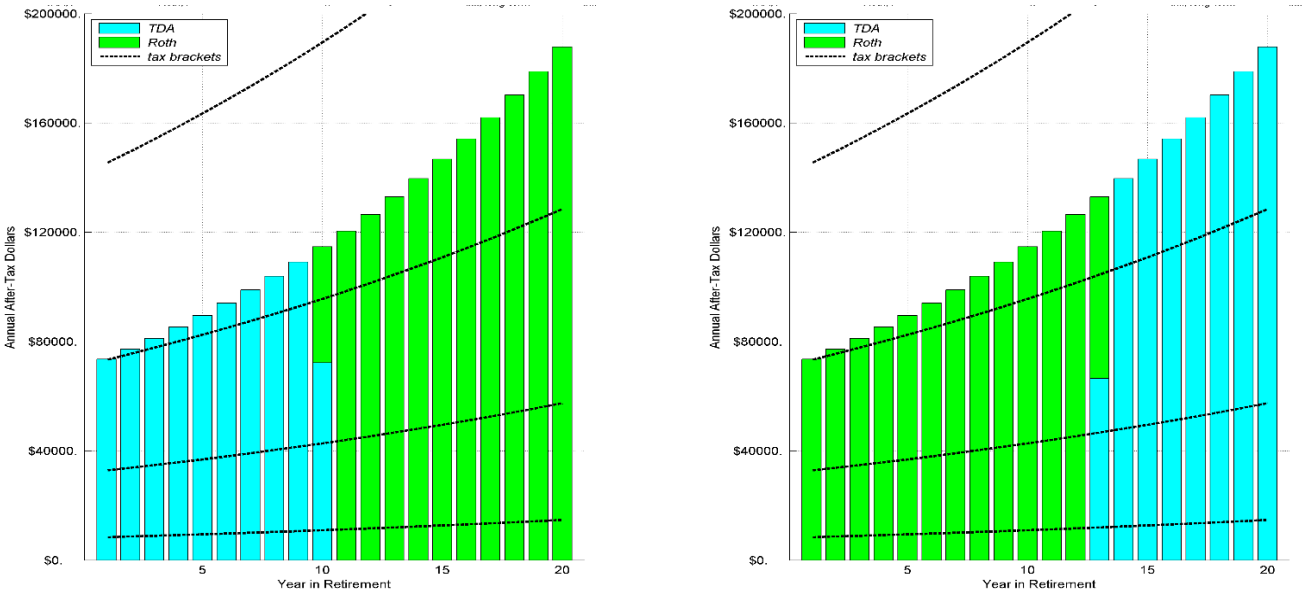
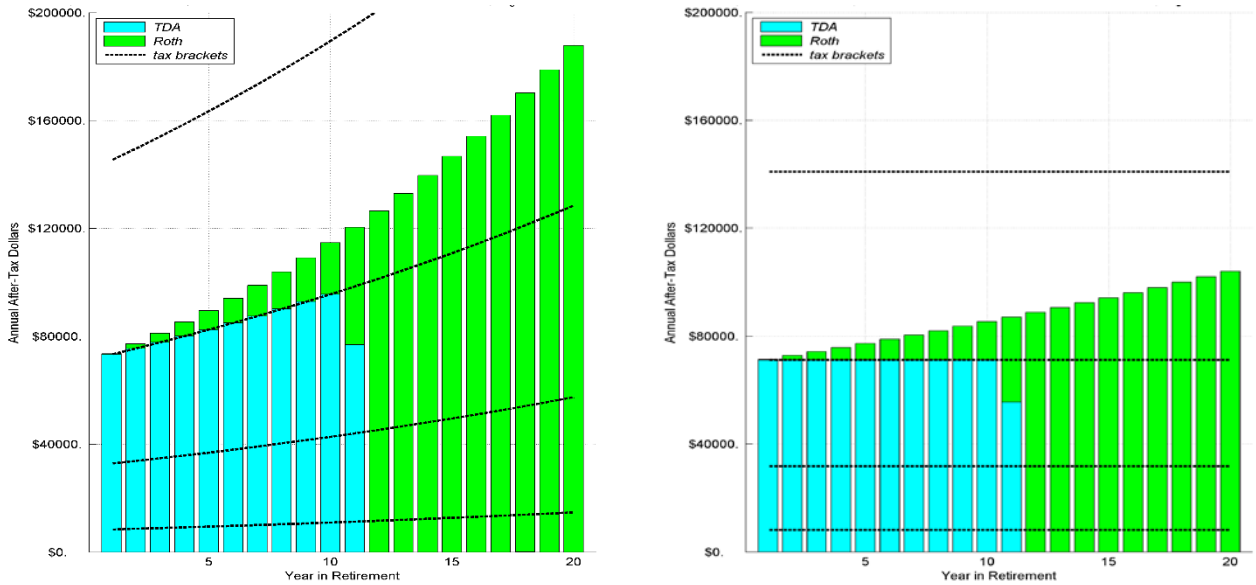
Figure 1: The Two Naïve Strategies.



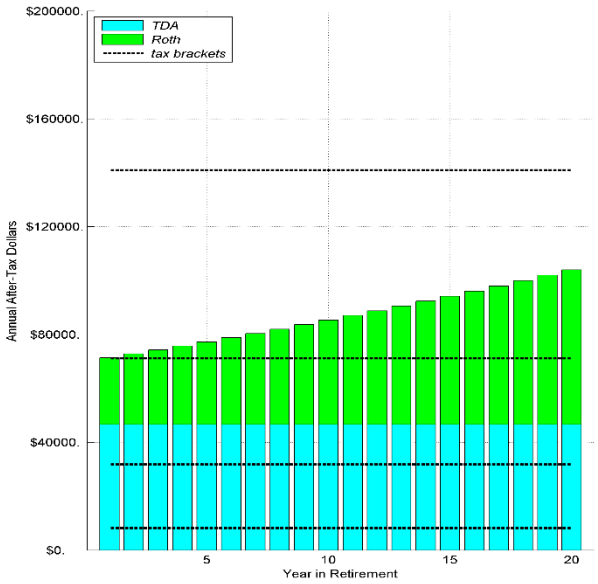
Notes: The dashed curves represent the tops of the IRS's 2014 tax brackets for single filers on an after-tax basis. The height of each column represents the after-tax consumption need in nominal dollars in the year corresponding to the column. Each column is divided into blue for consumption satisfied by the TDA account and green for consumption satisfied by the Roth account. In the left panel of Figure 1, the TDA account is completely drained and then the Roth account is used. In the right panel of Figure 1, the Roth account is completely drained before the TDA account is used. The specific model parameters used in Figures 1–3 are given in the caption of Table 1.

Figure 2: The Best Informed Strategy.



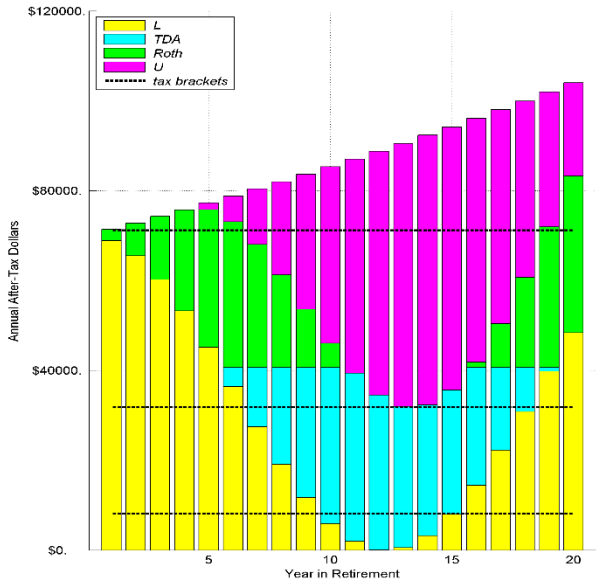
Notes: The left panel is in nominal dollars, and the right panel is in real dollars. The tax brackets, denoted by the dashed curves in the left panel, become constant over time in the right panel's real dollar perspective. The specific model parameters are given in the caption of Table 1 below.

Figure 3: An Exact, Optimal Strategy.



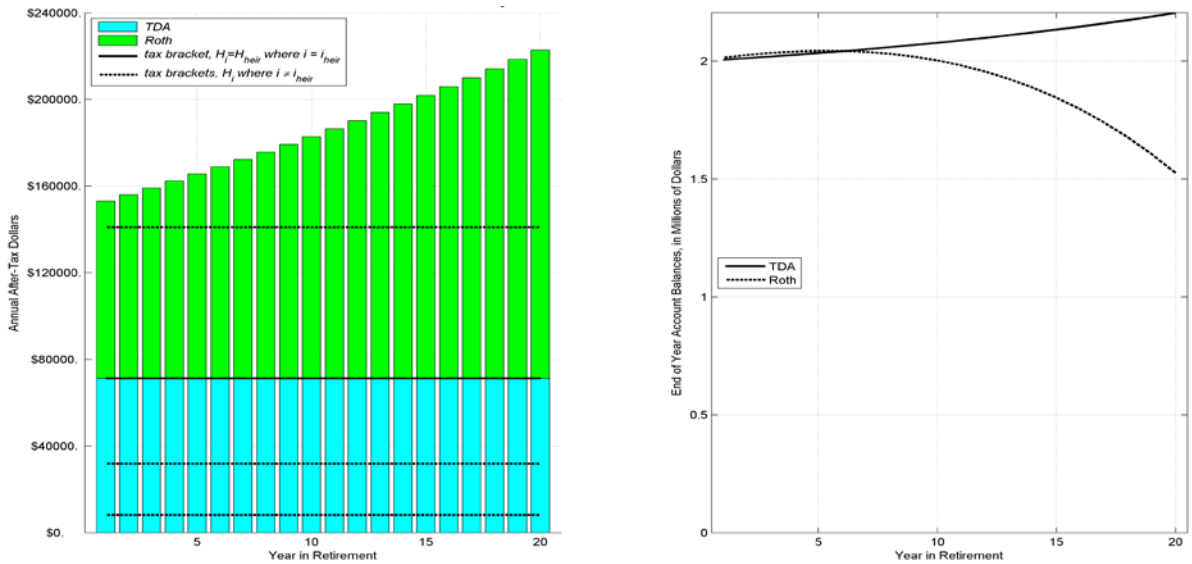
Notes: The optimal strategy shown above is straightforward: the investor withdraws TDA money up to an amount that is constant in real dollars over time. The specific model parameters are given in the caption in Table 1.

Figure 4: An Exact, Optimal Strategy in the Presence of Other Sources of Income.



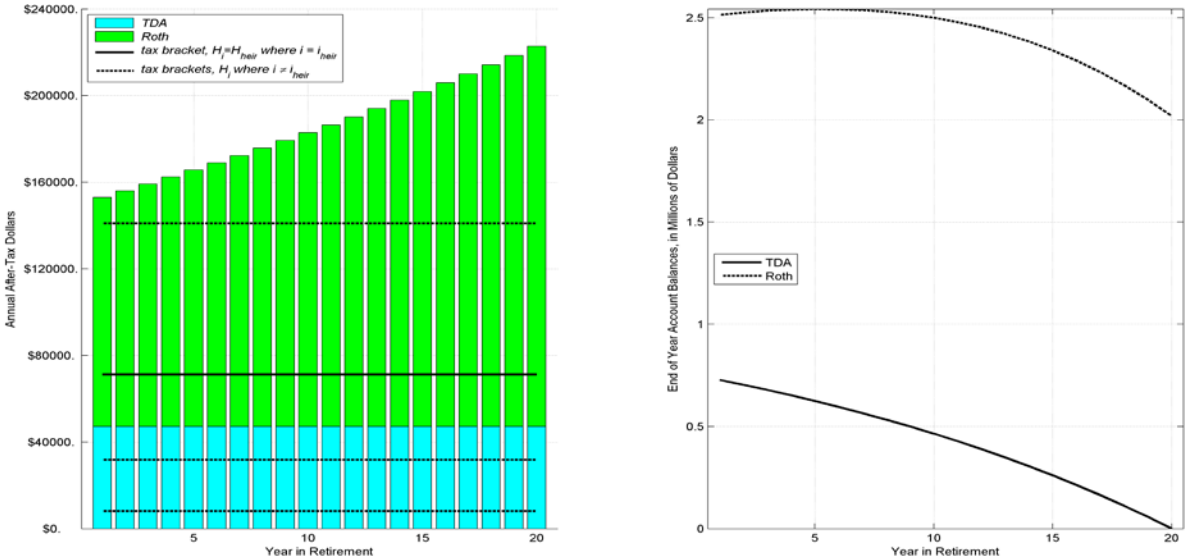
Notes: Sources other than the TDA that are taxable as income are labeled L and shown in yellow. Sources other than the Roth that are not taxed as income are labelled U and shown in magenta. At t = 0, the TDA account starts with $250,000 and the Roth account starts with $800,000. All other model parameters are given in the caption in Table 1.

Figure 5: Case a — Optimal Withdrawals Lead to both TDA and Roth Bequests.



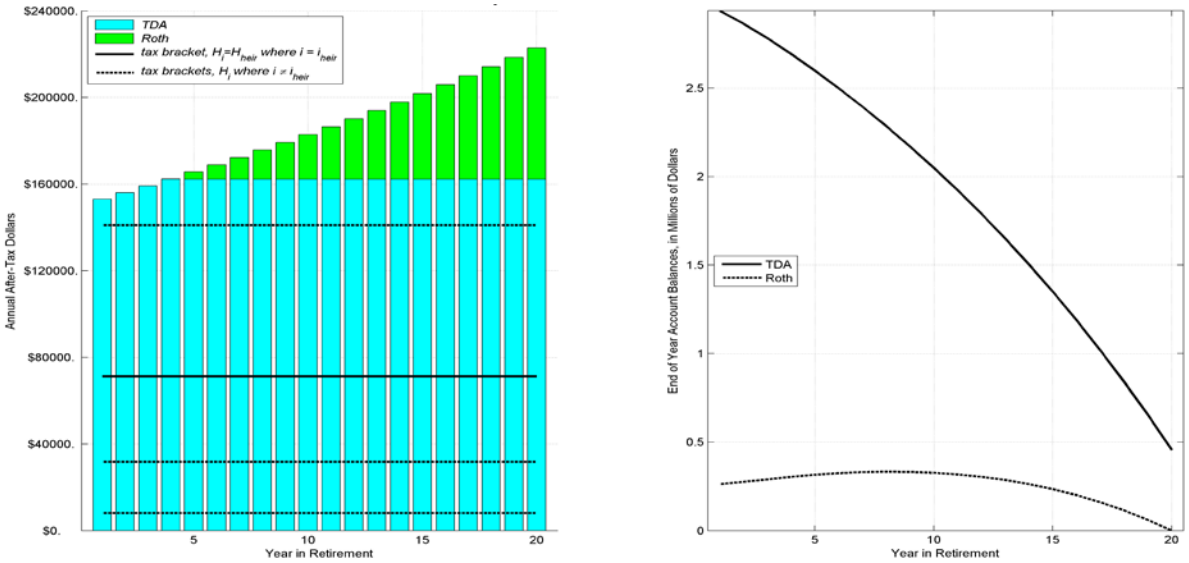
Notes: The panel on the right shows the balances remaining in the TDA and Roth accounts over time. Neither account is exhausted at time t_{death}. The tax rate to the heir is assumed to be 25%. This corresponds to the value of H_{heir} denoted by the solid black horizontal line in the left panel. We assume both the TDA and Roth accounts start with $2,000,000 and grow at a real annual rate of 5%. The consumption needs start at $150,000 and then grow at a real annual rate of 2% to model increasing medical costs.

Figure 6: Case b — Optimal Withdrawals Lead to a Roth Bequest, but No TDA Bequest.



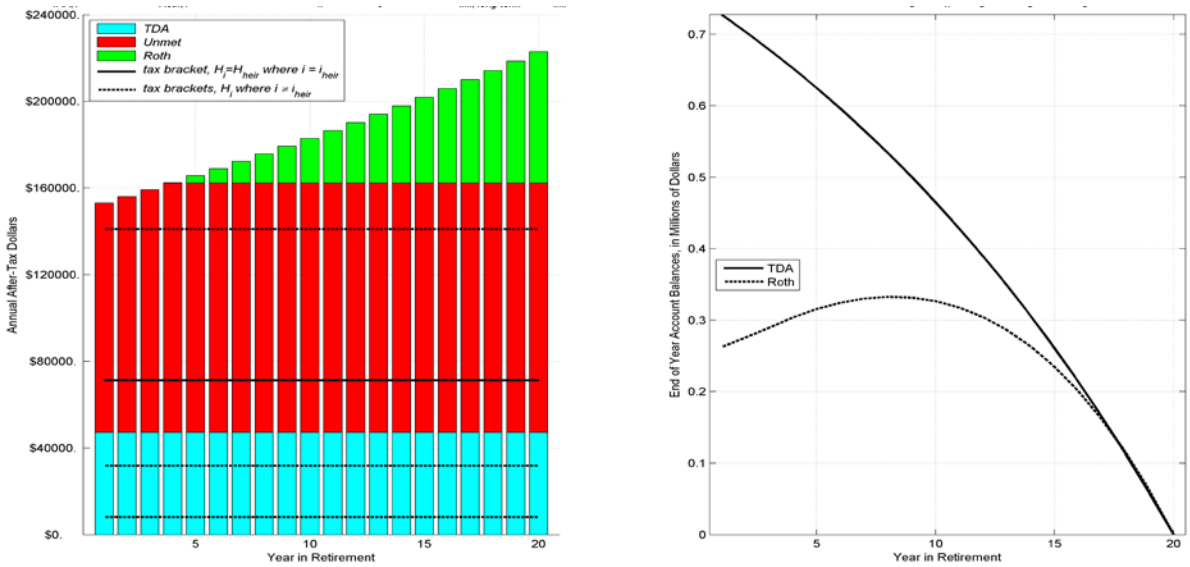
Notes: In this example the TDA starts with three quarters of a million dollars and the Roth starts with 2.5 million dollars. All other parameters are identical to those used in Figure 5.

Figure 7: Case c — Optimal Withdrawals Lead to a TDA Bequest, but No Roth Bequest.



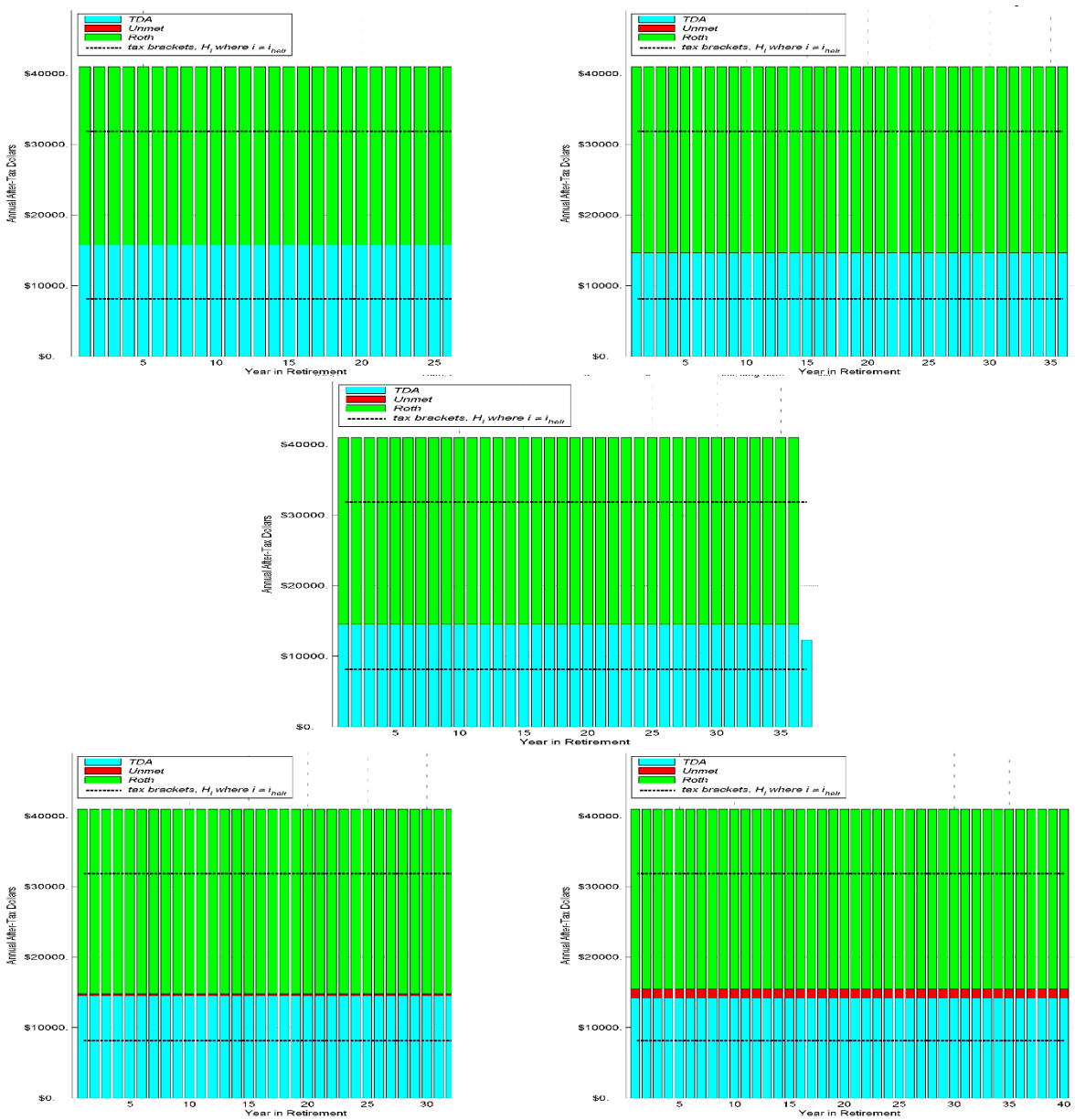
Notes: In this example the TDA starts with 3 million dollars and the Roth starts with a quarter of a million dollars. All other parameters are identical to those used in Figure 5.

Figure 8: Case d — Optimal Withdrawals Lead to neither a TDA nor a Roth Bequest.



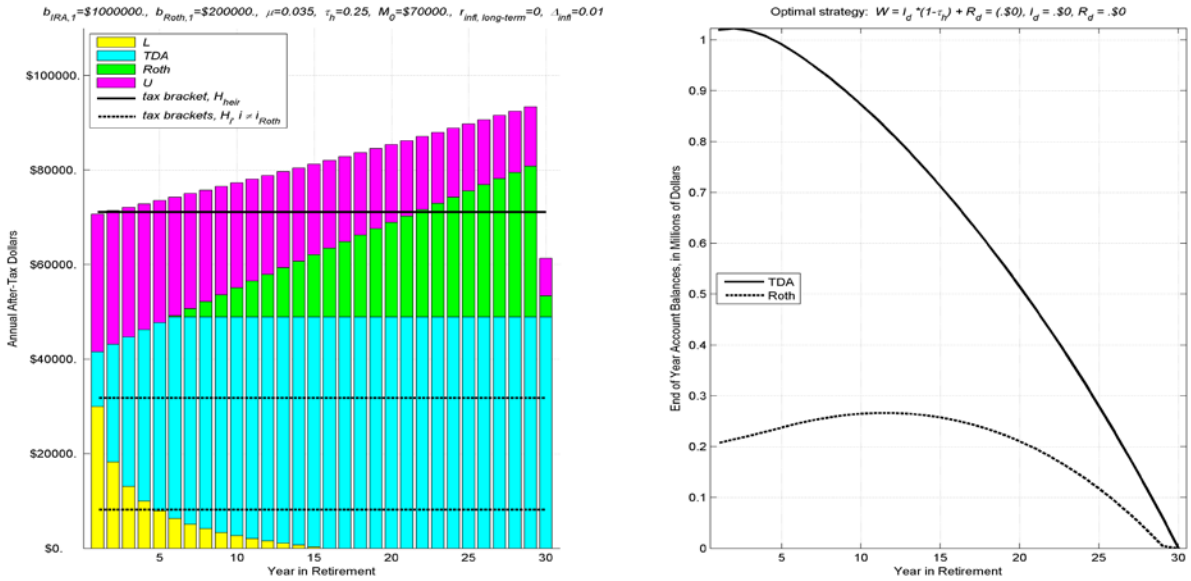
Notes: In this case, it is not possible for the consumption desires of the retiree to be met. The unmet consumption needs are shown in red. In this example the TDA starts with three quarters of a million dollars and the Roth starts with a quarter of a million dollars. All other parameters are identical to those used in Figure 5.

Figure 9: Determining Optimal Portfolio Longevity and an Optimal Strategy.



Notes: The upper left panel, t_{death} = 30 years, leaves about \$225,000 in real (t=0) Roth dollars at death. This is reduced to about \$12,000 in the upper right panel, t_{death} = 36. On the other hand, the lower right panel, t_{death} = 40, leaves a gap (in red) of \$1,333 real dollars every year. This gap is reduced to \$283 in the lower left panel, t_{death} = 37. The center panel corresponds to the optimal longevity where the gap disappears and there is no left over money at death. In this panel, α = 0.30, so the height of the last column is 0.30 as tall as the full year's height would be, and the optimal longevity is established to be 36.30 years. The graph shows this optimal longevity can be obtained by consuming \$26,500 real after-tax TDA dollars and \$14,500 real Roth dollars in each of the first 36 years and only TDA dollars in the 0.30 × 12 = 3.6 months of the 37th year.

Figure 10: Optimal Longevity with Sources of Income Other than the TDA and Roth Accounts.



Notes: In this example, the retiree starts with a $1,000,000 TDA account and a $200,000 Roth account. Both accounts grow at a real annual rate of 3.5%. The consumption needs of the retiree start at $70,000 and grow at a real annual rate of 1% to reflect increasing medical costs. $U(t)$ (in magenta) corresponds to an annuity that is not taxed as income. This annuity is assumed to pay $30,000 nominal after-tax dollars every year, which corresponds to a real annual rate of -3% to account for inflation. $L(t)$ (in yellow) corresponds to income due to part time work, which diminishes over the first 16 years of retirement. The optimal strategy shown in the figure corresponds to satisfying consumption needs not addressed by the annuity or part time work by using TDA money in each year up to a total after-tax income of $49,000 (including the part time work) and using Roth money for any remaining consumption needs. This leads to the optimal portfolio longevity for the retiree, which is 29.65 years.

Table 1: Comparison of the Two Naïve Strategies, the Best Informed Strategy, and Our Optimal Strategy.

Strategy	Associated Figure	After Tax Bequest in Real Dollars	Percent Below Optimal
Naïve: IRA, then Roth	Figure 1a	$521,867.50	10%
Naïve: Roth, then IRA	Figure 1b	$428,790.21	26%
Best Informed (up to 25% tax bracket)	Figure 2	$532,117.96	8%
Optimal	Figure 3	$577,724.52	0%

Notes: The table is built from the following model parameters used to generate Figures 1–3: The tax brackets and rates in year $t = 0$ are from the 2014 IRS tax code for a single filer. The brackets increase by the rate of inflation, which we set to 3%. Both the TDA and Roth accounts start with $800,000 dollars at $t = 0$ and grow at an annual real rate of 4%. The consumption needs of the retiree are $70,000 in the first year and grow at an annual real rate of 2% to approximate rising medical costs. The number of years before the retiree dies and leaves a bequest to the heir is assumed to be $t_{\text{death}} = 20$, and the marginal tax rate for the heir is assumed to be 33%. The bequest to the heir, which is expressed in real ($t = 0$) dollars, is computed by adding the after-tax worth of the TDA to the worth of the Roth at time t_{death}. In nominal dollars, the difference in the optimal bequest and the bequest from the best informed strategy is $(\$577{,}724.52 - \$532{,}117.96)(1.03)^{20} = \$82{,}370.52$.

Table 2: Comparison, for the Case Used in Figure 9, of the Portfolio Longevity Using the Two Naïve Strategies, the Best Informed Strategy, and the Optimal Strategy.

Strategy	Longevity (years)
Naïve: IRA, then Roth	34.71
Naïve: Roth, then IRA	34.69
Best Informed (up to 10% tax bracket)	35.78
Optimal	36.30

Notes: The best informed strategy beats both naïve strategies by slightly over a year, but is outperformed by our optimal strategy by just over half a year. The best informed strategy corresponds to spending TDA money only up to the top of the 10% tax bracket for as long as possible. The longevity of the best informed strategy, 35.78 years, is slightly longer than the longevity of the next best informed strategy, 35.71 years, where TDA money is spent up to the top of the 15% tax bracket.